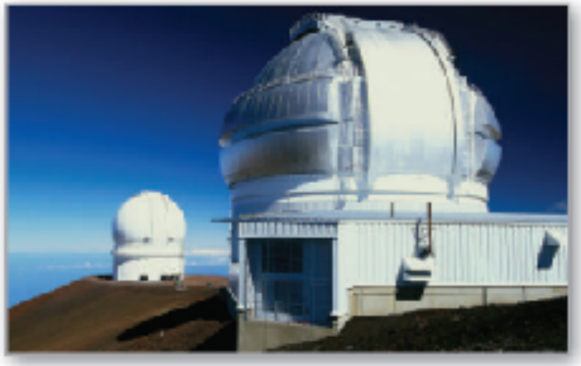
P Preparation for Calculus



P.2 Linear Models and Rates of Change

Objectives

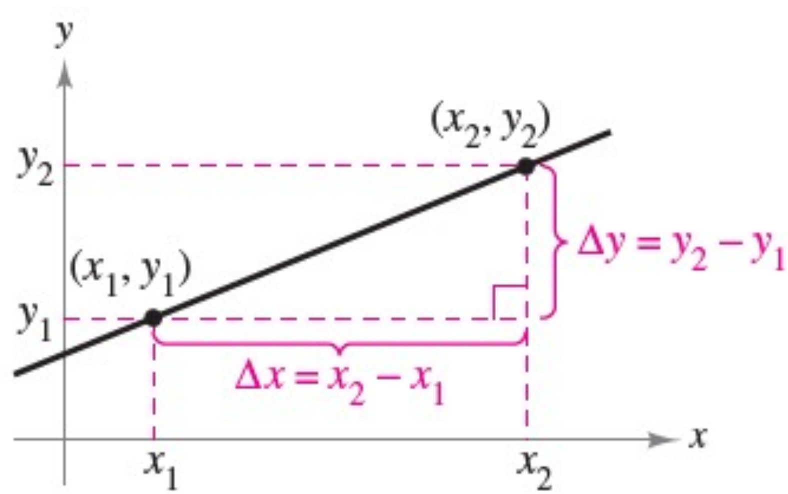
- Find the slope of a line passing through two points.
- Write the equation of a line with a given point and slope.
- Interpret slope as a ratio or as a rate in a real-life application.
- Sketch the graph of a linear equation in slope-intercept form.
- Write equations of lines that are parallel or perpendicular to a given line.

The Slope of a Line

The Slope of a Line

The **slope** of a nonvertical line is a measure of the number of units the line rises (or falls) vertically for each unit of horizontal change from left to right.

Consider the two points (x_1, y_1) and (x_2, y_2) on the line in Figure P.12.



$\Delta y = y_2 - y_1 = \text{change in } y$

$\Delta x = x_2 - x_1 = \text{change in } x$

Figure P.12

The Slope of a Line

As you move from left to right along this line, a vertical change of

$$\Delta y = y_2 - y_1 \qquad \text{Change in } y$$

units corresponds to a horizontal change of

$$\Delta x = x_2 - x_1 \qquad \text{Change in } x$$

units. (The symbol Δ is the uppercase Greek letter *delta*, and the symbols Δy and Δx are read "delta y" and "delta x.")

The Slope of a Line

Definition of the Slope of a Line

The **slope** m of the nonvertical line passing through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

Slope is not defined for vertical lines.

The Slope of a Line

When using the formula for slope, note that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}.$$

So, it does not matter in which order you subtract *as long as* you are consistent and both “subtracted coordinates” come from the same point.

The Slope of a Line

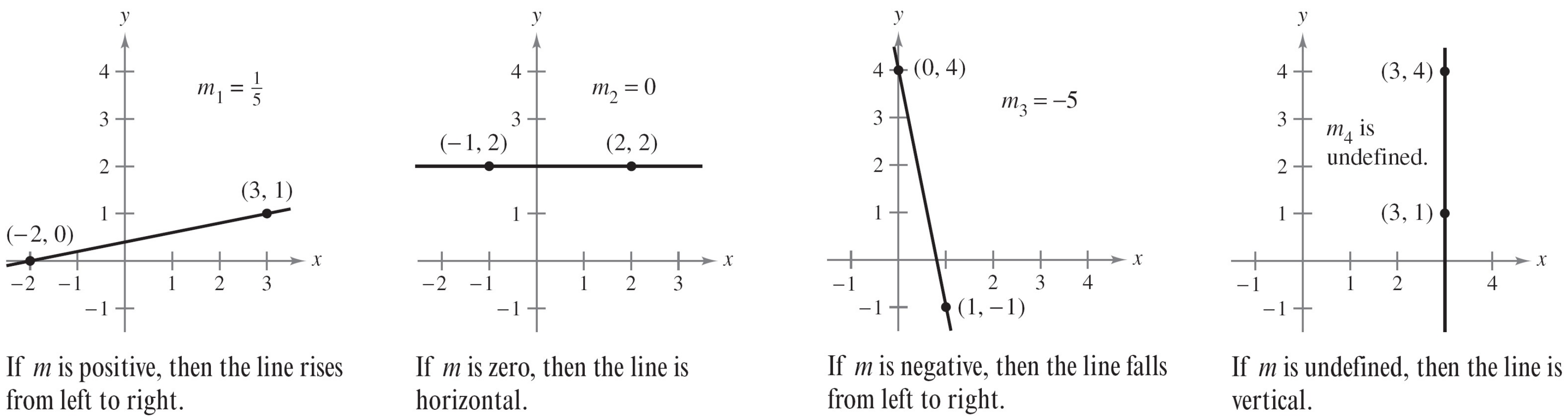


Figure P.13

Figure P.13 shows four lines: one has a positive slope, one has a slope of zero, one has a negative slope, and one has an "undefined" slope.

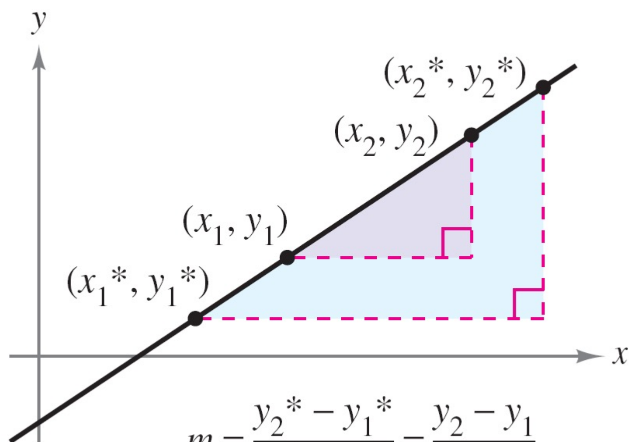
In general, the greater the absolute value of the slope of a line, the steeper the line.

Equations of Lines

Equations of Lines

Any two points on a nonvertical line can be used to calculate its slope.

This can be verified from the similar triangles shown in Figure P.14.



$$m = \frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}$$

Any two points on a nonvertical line can be used to determine its slope.

Figure P.14

Equations of Lines

If (x_1, y_1) is a point on a nonvertical line that has a slope of m and (x, y) is *any other* point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation in the variables x and y can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

which is called the **point-slope form** of the equation of a line.

Equations of Lines

Point-Slope Form of the Equation of a Line

The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is

$$y - y_1 = m(x - x_1).$$

Example 1 – *Finding an Equation of a Line*

Find an equation of the line that has a slope of 3 and passes through the point (1, –2). Then sketch the line.

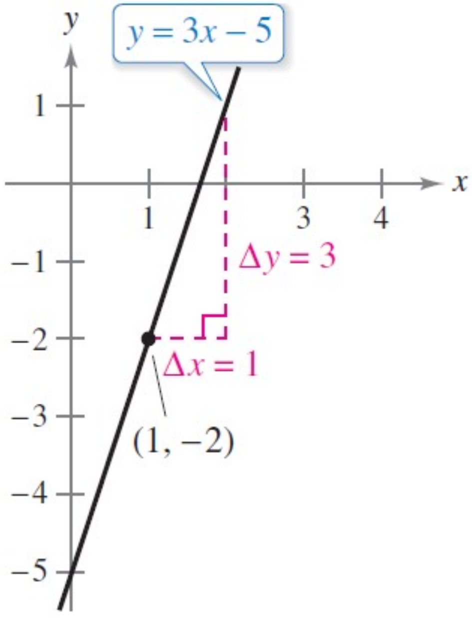
Solution:

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute -2 for y_1, 1 for x_1, and 3 for m.
$y + 2 = 3x - 3$	Simplify.
$y = 3x - 5$	Solve for y.

Example 1 – *Solution*

cont'd

To sketch the line, first plot the point (1, –2). Then, because the slope is m = 3, you can locate a second point on the line by moving one unit to the right and three units upward, as shown in Figure P.15.



The line with a slope of 3 passing through the point $(1, -2)$

Figure P.15

Ratios and Rates of Change

Ratios and Rates of Change

The slope of a line can be interpreted as either a *ratio* or a *rate*.

If the x- and y-axes have the same unit of measure, then the slope has no units and is a **ratio.**

If the x- and y-axes have different units of measure, then the slope is a rate or **rate of change.**

Example 2 – *Using Slope as a Ratio*

The maximum recommended slope of a wheelchair ramp is 1/12. A business installs a wheelchair ramp that rises to a height of 22 inches over a length of 24 feet, as shown in Figure P.16. Is the ramp steeper than recommended?

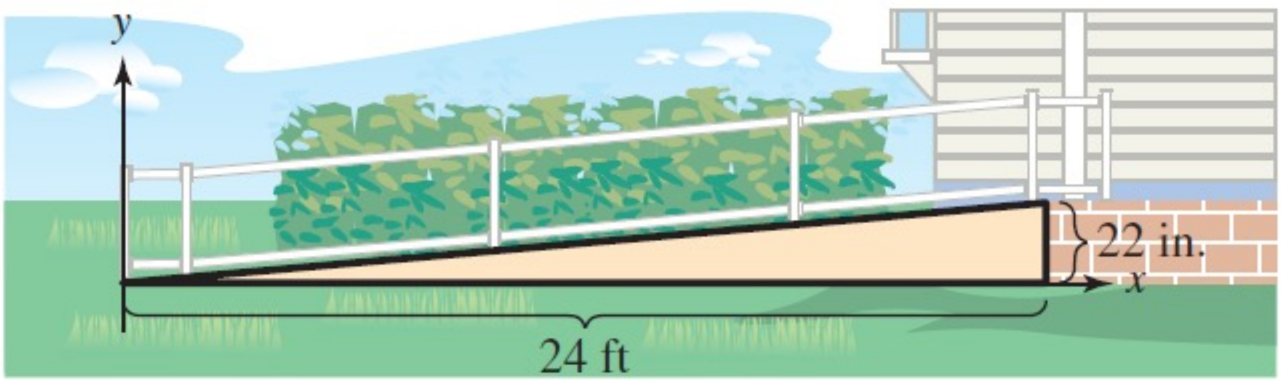


Figure P.16

Example 2 – *Solution*

The length of the ramp is 24 feet or 12 (24) = 288 inches. The slope of the ramp is the ratio of its height (the rise) to its length (the run).

$$\text{Slope of ramp} = \frac{\text{rise}}{\text{run}}$$

$$= \frac{22 \text{ in.}}{288 \text{ in.}}$$

$$\approx 0.076$$

Because the slope of the ramp is less than ½ ≈ 0.083, the ramp is not steeper than recommended. Note that the slope is a ratio and has no units.

Example 3 – *Using Slope as a Rate of Change*

The population of Oregon was about 3,831,000 in 2010 and about 3,970,000 in 2014. Find the average rate of change of the population over this four-year period. What will the population of Oregon be in 2024?

Solution:

Over this four-year period, the average rate of change of the population of Oregon was

$$\text{Rate of change} = \frac{\text{change in population}}{\text{change in years}}$$

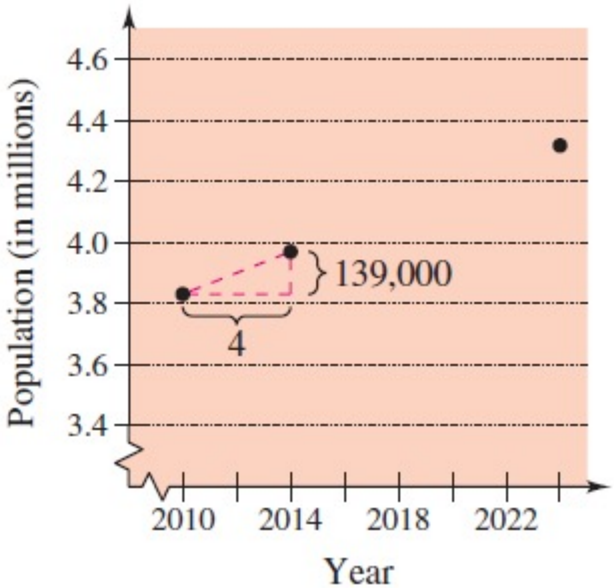
Example 3 – *Solution*

cont'd

$$= \frac{3{,}970{,}000 - 3{,}831{,}000}{2014 - 2010}$$

$$= 34{,}750 \text{ people per year.}$$

Assuming that Oregon's population continues to increase at this same rate for the next 10 years, it will have a 2024 population of about 4,318,000.
(See Figure P.17.)



Population of Oregon

Figure P.17

Ratios and Rates of Change

The rate of change found in Example 3 is an **average rate of change**. An average rate of change is always calculated over an interval.

Graphing Linear Models

Graphing Linear Models

Many problems in analytic geometry can be classified in two basic categories:

1. Given a graph (or parts of it), find its equation.

2. Given an equation, sketch its graph.

For lines, problems in the first category can be solved by using the point-slope form. The point-slope form, however, is not especially useful for solving problems in the second category.

Graphing Linear Models

The form that is better suited to sketching the graph of a line is the **slope-intercept** form of the equation of a line.

The Slope-Intercept Form of the Equation of a Line

The graph of the linear equation

$$y = mx + b \qquad \text{Slope-intercept form}$$

is a line whose slope is m and whose y-intercept is $(0, b)$.

Example 4 – *Sketching Lines in the Plane*

Sketch the graph of each equation.

a. $y = 2x + 1$

b. $y = 2$

c. $3y + x – 6 = 0$

Example 4(a) – *Solution*

cont'd

Because b = 1, the y-intercept is (0, 1).

Because the slope is m = 2, you know that the line rises two units for each unit it moves to the right, as shown in Figure P.18(a).

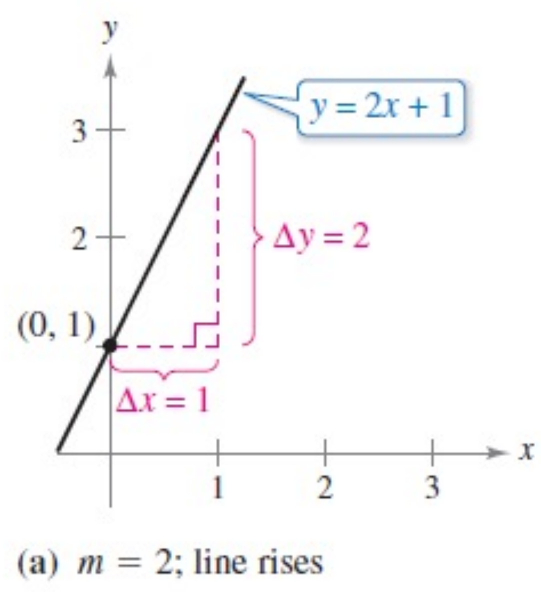


Figure P.18(a)

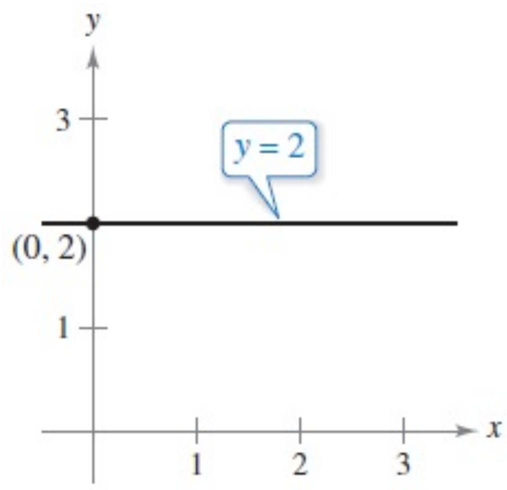
Example 4(b) – *Solution*

cont'd

By writing the equation $y = 2$ in slope-intercept form

$$y = (0)x + 2$$

you can see that the slope is $m = 0$ and the y-intercept is (0,2). Because the slope is zero, you know that the line is horizontal, as shown in Figure P.18(b).



(b) $m = 0$; line is horizontal

Figure P.18(b)

Example 4(c) – *Solution* cont'd

Begin by writing the equation in slope-intercept form.

$$3y + x - 6 = 0 \qquad \text{Write original equation.}$$

$$3y = -x + 6 \qquad \text{Isolate } y\text{-term on the left.}$$

$$y = -\frac{1}{3}x + 2 \qquad \text{Slope-intercept form}$$

In this form, you can see that the *y*-intercept is (0, 2) and the slope is $m = -\frac{1}{3}$. This means that the line falls one unit for every three units it moves to the right.

Example 4(c) – *Solution*

cont'd

This is shown in Figure P.18(c).

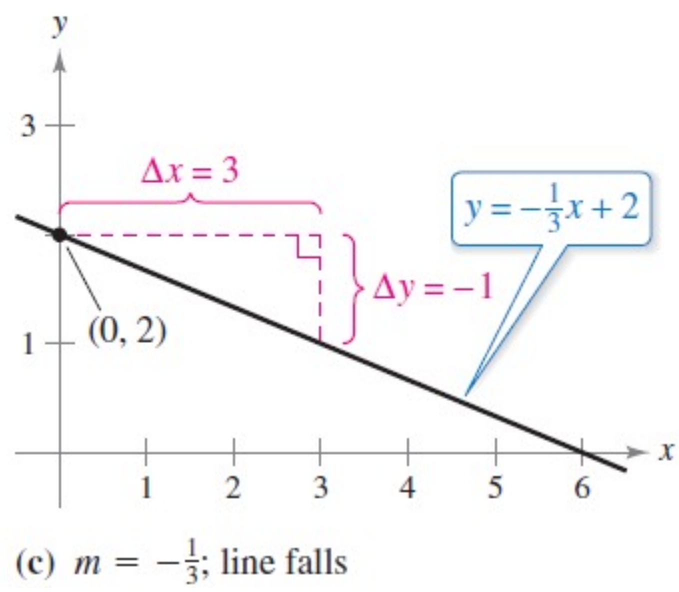


Figure P.18(c)

Graphing Linear Models

Because the slope of a vertical line is not defined, its equation cannot be written in the slope-intercept form. However, the equation of any line can be written in the **general form**

$$Ax + By + C = 0 \qquad \text{General form of the equation of a line}$$

where *A* and *B* are not *both* zero. For instance, the vertical line

$$x = a \qquad \text{Vertical line}$$

can be represented by the general form

$$x - a = 0. \qquad \text{General form}$$

Graphing Linear Models

SUMMARY OF EQUATIONS OF LINES

1.	General form:	$Ax + By + C = 0$
2.	Vertical line:	$x = a$
3.	Horizontal line:	$y = b$
4.	Slope-intercept form:	$y = mx + b$
5.	Point-slope form:	$y - y_1 = m(x - x_1)$

Parallel and Perpendicular Lines

Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular, as shown in Figure P.19.

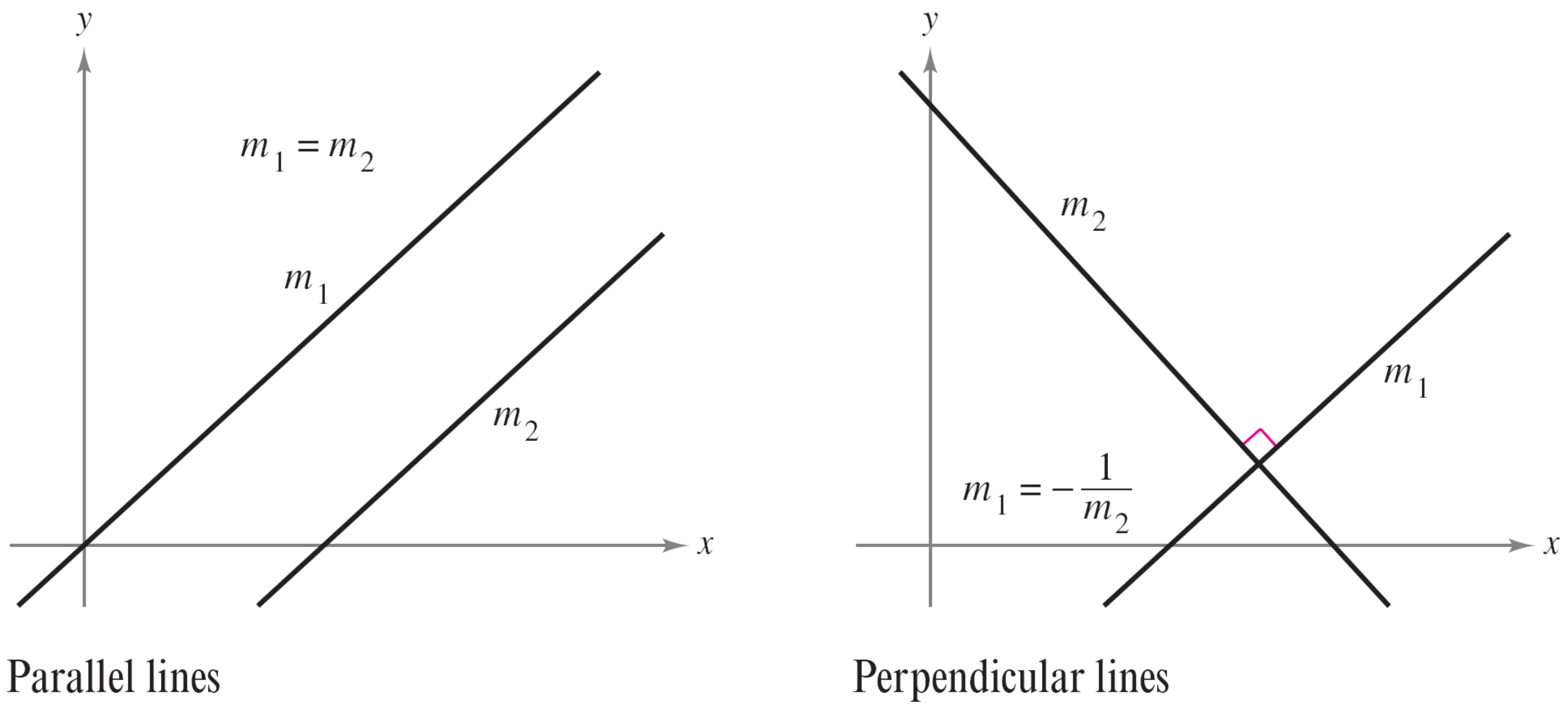


Figure P.19

Parallel and Perpendicular Lines

Parallel and Perpendicular Lines

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal—that is, if and only if

   $$m_1 = m_2.$$

   Parallel ⟺ Slopes are equal.

2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other—that is, if and only if

   $$m_1 = -\frac{1}{m_2}.$$

   Perpendicular ⟺ Slopes are negative reciprocals.

Example 5 – *Finding Parallel and Perpendicular Lines*

Find the general forms of the equations of the lines that pass through the point (2, –1) and are

(a) parallel to the line
$2x - 3y = 5$

(b) perpendicular to the line
$2x - 3y = 5.$

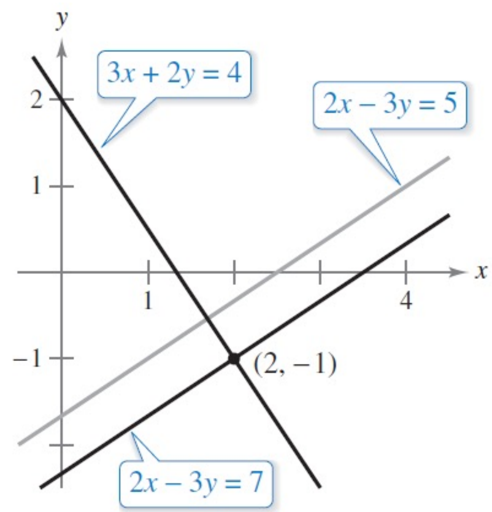
Example 5 – *Solution*

Begin by writing the linear equation $2x - 3y = 5$ in slope-intercept form.

$$2x - 3y = 5 \qquad \text{Write original equation.}$$

$$y = \frac{2}{3}x - \frac{5}{3} \qquad \text{Slope-intercept form}$$

So, the given line has a slope of $m = \frac{2}{3}$. (See Figure P.20.)



Lines parallel and perpendicular to $2x - 3y = 5$

Figure P.20

Example 5 – *Solution*

cont'd

a. The line through (2, –1) that is parallel to the given line also has a slope of 2/3.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$

$$y - (-1) = \tfrac{2}{3}(x - 2) \quad \text{Substitute.}$$

$$3(y + 1) = 2(x - 2) \quad \text{Simplify.}$$

$$3y + 3 = 2x - 4 \quad \text{Distributive Property}$$

$$2x - 3y - 7 = 0 \quad \text{General form}$$

Note the similarity to the equation of the given line, $2x - 3y = 5$.

Example 5 – *Solution*

cont'd

b. Using the negative reciprocal of the slope of the given line, you can determine that the slope of a line perpendicular to the given line is –3/2.

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - (-1) = -\tfrac{3}{2}(x - 2) \qquad \text{Substitute.}$$

$$2(y + 1) = -3(x - 2) \qquad \text{Simplify.}$$

$$2y + 2 = -3x + 6 \qquad \text{Distributive Property}$$

$$3x + 2y - 4 = 0 \qquad \text{General form}$$